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Characteristics of a pure-state ambiguity function

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Abstract
We present the necessary and sufficient condition for a square integrable function on $\mathbb{R}^{2N}$ to be an ambiguity function corresponding to a square integrable function on $\mathbb{R}^N$. This condition has the form of an integral equation. We also list some easier to check necessary conditions that must be fulfilled by a function that is an ambiguity function of a pure state. We show how to construct a wavefunction corresponding to a given ambiguity function and we present examples of how our formal results can be used in practice.

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1. Introduction

The ambiguity function [1], known also as a characteristic function [2] or Shirley function [3], is used to describe continuous variables physical systems. Initially, it was introduced for classical systems in radar theory [1], and then in laser spectroscopy [3, 4], quantum optics [5–7] and for analysis of the dynamics of dissipative systems [8]. For continuous variables, the descriptions of a state by its wavefunction (density operator) or its ambiguity function are completely equivalent. This function captures transparently the off-diagonal elements of the density matrix and thus is especially useful, e.g. for studying decoherence. Traditionally, investigations start from specifying the density operator and only then is the corresponding ambiguity function calculated. Then, one can easily check whether the state of a system is pure or mixed. However, sometimes it is more convenient to start the description of the state directly with its ambiguity function. For example, dealing with a pure-state ambiguity function[5] one may modify this function slightly—the change may originate from phenomenologically or empirically suggested adaptations of formalism. Then, it may not be immediately clear whether the modified function is also a pure-state ambiguity function or not. Thus, a method of recognizing the pure-state ambiguity functions would be useful. It is the purpose of this

[5] From now on we shall call an ambiguity function corresponding to a pure state, a 'pure-state ambiguity function'.

paper to show that there exists a condition necessary and sufficient for a function to be such a pure-state ambiguity function.

In physics, a pure state is usually defined as a square integrable function normalized to 1. However, the ambiguity function can be calculated for any square integrable function, and from a mathematical point of view, it is easier and more elegant to consider an entire space of square integrable functions and then a set of normalized functions as a subset of this space. In this paper, we shall focus on the following problem: is a given square integrable function defined on $\mathbb{R}^{2N}$ an ambiguity function corresponding to a square integrable function defined on $\mathbb{R}^N$ or not. Although to simplify the notation[6] we describe the case when $N = 1$, a generalization to any $N = 2, 3, 4, \ldots$ can be done without serious difficulties. For the convenience of the readers less familiar with mathematical notation and methods used in this paper, we include the appendix 'definitions and notation' with brief lists of definitions of symbols used. Further references to relevant textbooks are also included there.

Although this paper is predominantly of mathematical nature, the results presented are also of physical interest, because they provide tools to construct and investigate models related to realistic physical systems. Part of the possible applications comes from the fact that ambiguity function is convenient for studying open systems because the master equation has a quite simple form in this representation [8]. Our results allow us to perform the whole analysis of such systems using ambiguity representation and at the same time maintain control on whether subsystems are still pure or not or whether they become pure again during the interaction. Whenever a system described corresponds to a pure state, one can retrieve its wavefunction explicitly based solely on the results of this paper. For mixed states a similar reconstruction would be more complicated but also very useful as it was shown that in ambiguity representation interesting phase-space effects (e.g. sub-Planck structures) appear even for mixed states [9].

2. The fundamental condition

To simplify notation, we introduce the following symbol:

$$W_S[\psi](p,x) := \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2\pi}\, \mathrm{e}^{-\mathrm{i}p\,\xi}\, \psi^\star\left(\xi - \frac{x}{2}\right) \psi\left(\xi + \frac{x}{2}\right), \tag{1}$$

for the ambiguity function corresponding to $\psi \in L^2(\mathbb{R})$. In this way, wavefunction dependence of the ambiguity function is indicated explicitly.

From [10] (proposition 3.6 (ii), theorem 3.7, corollary 3.16, proposition 3.17), we obtain the following fact valid for the group of real numbers $\mathbb{R}$.

Fact 1. For $W_S[\cdot]$ we have

1° $\forall\psi \in L^2(\mathbb{R}) : W_S[\psi] \in C_\infty(\mathbb{R}^2)$
2° $W_S : L^2(\mathbb{R}) \ni \psi \longrightarrow W_S[\psi] \in C_\infty(\mathbb{R}^2)$ is a continuous map.
3° $\forall\psi \in L^2(\mathbb{R}) : W_S[\psi] \in L^2(\mathbb{R}^2)$
4° $W_S : L^2(\mathbb{R}) \ni \psi \longrightarrow W_S[\psi] \in L^2(\mathbb{R}^2)$ is a continuous map.

Now we can formulate and prove the following fact.

Fact 2. For all $\psi \in L^2(\mathbb{R})$ and for all $x, x_0, p, p_0 \in \mathbb{R}$:

$$W_S[\psi]^\star(p_0 - p, x_0 - x)W_S[\psi](p_0 + p, x_0 + x) = \int_{\mathbb{R}^2} \frac{\mathrm{d}x'\,\mathrm{d}p'}{2\pi}\, \mathrm{e}^{\mathrm{i}p_0x' - \mathrm{i}x_0p'} W_S[\psi]^\star(p' - p, x' - x)W_S[\psi](p' + p, x' + x). \tag{2}$$

[6] Mainly normalizing factors.

For $\psi \in \mathcal{S}(\mathbb{R})$, equation (2) can be checked by a direct calculation using standard methods of tempered distributions theory. Next, using the fact that a subspace $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we prove equation (2) for any function $\psi \in L^2(\mathbb{R})$. To avoid technicalities, we shall present only the main points of this proof. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of $\psi_n \in \mathcal{S}(\mathbb{R})$ convergent to $\psi \in L^2(\mathbb{R})$, i.e. $\lim_{n\to\infty} \|\psi - \psi_n\|_2 = 0$. According to fact 1 (2° and 4°), we have

$$\lim_{n\to\infty} \|W_S[\psi] - W_S[\psi_n]\|_\infty = 0, \qquad \text{and} \qquad \lim_{n\to\infty} \|W_S[\psi] - W_S[\psi_n]\|_2 = 0. \tag{3}$$

From this we conclude that there exist constants $M_2 > 0$ and $M_\infty > 0$ that satisfy inequalities

$$\forall n \in \mathbb{N} : \|W_S[\psi_n]\|_\infty < M_\infty \qquad \text{and} \qquad \|W_S[\psi_n]\|_2 < M_2.$$

Taking into account also the following estimates,

$$|W_S[\psi_n]^\star(p_0 - p, x_0 - x) W_S[\psi_n](p_0 + p, x_0 + x) - W_S[\psi]^\star(p_0 - p, x_0 - x) \\ \times W_S[\psi](p_0 + p, x_0 + x)| \leqslant 2\max(\|W_S[\psi]\|_\infty, M_\infty)||W_S[\psi_n] - W_S[\psi]||_\infty$$

and

$$\left| \int_{\mathbb{R}^2} \frac{\mathrm{d}p'\,\mathrm{d}x'}{2\pi}\, \mathrm{e}^{\mathrm{i}p_0 x' - \mathrm{i}p' x_0} [W_S[\psi_n]^\star(p' - p, x' - x) W_S[\psi_n](p' + p, x' + x) \right. \\ \left. - W_S[\psi]^\star(p_0 - p, x_0 - x) W_S[\psi](p' + p, x' + x)] \right| \\ \leqslant \max(\|W_S[\psi]\|_2, M_2)||W_S[\psi_n] - W_S[\psi]||_2\,,$$

recalling limits (3) and the density of space $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$, we check equation (2) for any $\psi \in L^2(\mathbb{R}^2)$.

Remark 1. Equation (2) is analogous to equation (3.8) from [11] that is being fulfilled by the Wigner function.

As we already know, for $\psi \in L^2(\mathbb{R})$, the ambiguity function is a square integrable function, it satisfies equation (2), and a condition $2\pi W_S[\psi](0, 0) = \|\psi\|_2^2 \geqslant 0$, [9]. It is interesting to establish whether any function satisfying equation (2) must be an ambiguity function of some ψ. To emphasize the crucial role of the equation of the form of (2) in our further considerations, we introduce the following definition.

Definition 1. *We say that $A \in L^2(\mathbb{R}^2)$ fulfils a fundamental condition if and only if for all $x, x_0, p, p_0 \in \mathbb{R}$ the following equation holds:*

$$A^\star(p_0 - p, x_0 - x) A(p_0 + p, x_0 + x) \\ = \int_{\mathbb{R}^2} \frac{\mathrm{d}x'\,\mathrm{d}p'}{2\pi}\, \mathrm{e}^{\mathrm{i}p_0 x' - \mathrm{i}x_0 p'} A^\star(p' - p, x' - x) A(p' + p, x' + x). \tag{4}$$

Let us now establish some basic properties of functions that fulfil this fundamental condition.

3. Properties of functions fulfilling the fundamental condition

It is not difficult to obtain

Fact 3. For any $A \in L^2(\mathbb{R}^2)$ that fulfils the fundamental condition (definition 1, equation 4), and for $F^A_{(p,x)}(\cdot_1, \cdot_2)$ defined as $F^A_{(p,x)}(\cdot_1, \cdot_2) := A^\star(\cdot_1 - p, \cdot_2 - x) A(\cdot_1 + p, \cdot_2 + x)$ for any

$(p, x) \in \mathbb{R}^2$, the following conditions hold:

1°. $\forall\,(p_0, x_0) \in \mathbb{R}^2 : 2\pi|A(p_0, x_0)|^2 = \int_{\mathbb{R}^2} \mathrm{d}x'\,\mathrm{d}p'\,\mathrm{e}^{\mathrm{i}p_0x' - \mathrm{i}x_0p'}|A(p', x')|^2$.
2°. $2\pi|A(0, 0)|^2 = \|A\|_2^2$.
3°. $(A(0, 0) = 0) \Leftrightarrow (A = 0 \text{ in } L^2(\mathbb{R}^2))$.
4°. $\forall (p, x) \in \mathbb{R}^2 : A^\star(0, 0)A(p, x) = A(0, 0)A^\star(-p, -x)$.
5°. $\forall (p, x) \in \mathbb{R}^2 : (A(0, 0) \neq 0) \Rightarrow (A(p, x) = A(0, 0)A^\star(-p, -x)A^\star(0, 0)^{-1})$.
6°. $(A(0, 0) \in \mathbb{R}) \Rightarrow (\forall (p, x) \in \mathbb{R}^2 : A(p, x) = A^\star(-p, -x))$.
7°. $\forall (p, x) \in \mathbb{R}^2 : F^A_{(p,x)} \in C_\infty(\mathbb{R}^2)$.
8°. $A \in C_\infty(\mathbb{R}^2)$.
9°. $\forall (p, x) \in \mathbb{R}^2 : F^A_{(p,x)} \in L^2(\mathbb{R}^2)$.
10°. $A \in L^4(\mathbb{R}^2)$.

Since equation (4) is invariant under transformation $A \to \lambda A$, for $\lambda \in \mathbb{C}$, we conclude that not every solution of this equation fulfils condition $A(0, 0) \geqslant 0$, and so not every one is of the form $A = W_S[\psi]$. Nevertheless, condition $A(0, 0) \geqslant 0$ determines those solutions of equation (4) for which $A = W_S[\psi]$. For $A = 0$, we have $W_S[0] = 0$. In the considerations that follow it is, therefore, sufficient to assume that A is a nonzero element of $L^2(\mathbb{R}^2)$, i.e. $A \in (L^2(\mathbb{R}^2)\backslash\{0\})$. According to this assumption $A(\cdot, 0) \in (L^2(\mathbb{R})\backslash\{0\})$, and, since the Fourier transform is a unitary and consequently invertible map $L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, we find

$$\exists\tau \in \mathbb{R} : \int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{\mathrm{i}\tau p}A(p, 0) \neq 0. \tag{5}$$

Remark 2. If $A \in (L^2(\mathbb{R}^2)\backslash 0)$ then the set $\left\{\tau \in \mathbb{R} : \int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{\mathrm{i}\tau p}A(p, 0) \neq 0\right\}$ has a nonzero Lebesgue measure.

This allows us to introduce the following definition.

Definition 2. *Let τ be a fixed real number, such that inequality $\int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{\mathrm{i}\tau p}A(p, 0) \neq 0$ holds, and let $n_\tau \in (\mathbb{C}\backslash\{0\})$. We define a function ψ_{τ,n_τ} as*

$$\psi_{\tau,n_\tau} : \mathbb{R} \to \psi_{\tau,n_\tau}(x) := n_\tau \int_{\mathbb{R}} \mathrm{d}p\, \exp\left[\mathrm{i}\frac{x+\tau}{2}p\right]A(p, x-\tau) \in \mathbb{C}. \tag{6}$$

Remark 3. For any $\psi \in L^2(\mathbb{R})$ and $\tau \in \mathbb{R}$, the following inequality holds: $\int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{\mathrm{i}\tau p}W_S[\psi](p, 0) = |\psi(\tau)| \geqslant 0$.

The function (6) fulfils the condition $\psi_{\tau,n_\tau}(\tau) \neq 0$. Using equation (4), we find

$$0 \leqslant \|\psi_{\tau,n_\tau}\|_2^2 = 2\pi|n_\tau|^2A^\star(0, 0)\int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{\mathrm{i}\tau p}A(p, 0), \tag{7}$$

which, taking into account fact 3.3°, means that

$$A^\star(0, 0)\int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{\mathrm{i}\tau p}A(p, 0) > 0. \tag{8}$$

Moreover, from fact 3 for any $p, x \in \mathbb{R}$ we have

$$W_S[\psi_{\tau,n_\tau}](p, x) = \left\{|n_\tau|^2\left[\int_{\mathbb{R}} \mathrm{d}z\,\mathrm{e}^{\mathrm{i}\tau z}A(z, 0)\right]^\star\right\}A(p, x). \tag{9}$$

Connecting equations (8) and (9) we obtain the following fact.

Fact 4. For $A \in (L^2(\mathbb{R})\backslash\{0\})$ that fulfils the fundamental condition (4) and ψ_{τ,n_τ} defined by (6), the following four conditions are equivalent:

1° $A(0,0) > 0$

2° $\int_{\mathbb{R}} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}\tau z} A(z,0) > 0$

3° $\exists n_\tau \in (\mathbb{C}\backslash\{0\}) : |n_\tau|^2 \left[\int_{\mathbb{R}} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}\tau z} A(z,0)\right]^\star = 1$

4° $A = W_S[\psi_{\tau,n_\tau}]$.

Finally, connecting fact 4.1°, 4.4° and a trivial solution for $A = 0$, we can formulate

Corollary 1. *If $A \in L^2(\mathbb{R})$ fulfils the fundamental condition, equation (4), then the following two conditions are equivalent*

1° $A(0,0) \geqslant 0$

2° $\exists \psi \in L^2(\mathbb{R}) : A = W_S[\psi]$.

Remark 4. If $A \in L^2(\mathbb{R})$ fulfils the fundamental condition, equation (4), then

$$\exists(\alpha,\psi) \in [0, 2\pi[\times L^2(\mathbb{R}) : A = \mathrm{e}^{\mathrm{i}\alpha} W_S[\psi].$$

This means that every solution of equation (4) is (up to a phase factor) an ambiguity function of a square integrable function on $\mathbb{R}$.

Corollary 2. *Assuming that for $A \in L^2(\mathbb{R}^2)$ the fundamental condition holds and $A(0,0) \neq 0$, then there exist $c \in (\mathbb{C}\backslash\{0\})$ and $\psi \in L^2(\mathbb{R})$ such that $\|\psi\|_2 = 1$ and $cA = W_S[\psi]$.*

4. Examples

As an illustration let us now consider some examples that exploit the results presented before. In [9] (section C.2), we have considered the wavefunction

$$\begin{aligned}
\psi(x) &= \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{n!}\right)^{\frac{1}{2}} (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(x) \exp\left(-\frac{x^2}{2}\right) \\
&= \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{x^2+\bar{n}}{2}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sqrt{\frac{\bar{n}}{2}}\right)^n H_n(x) \\
&= \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{x^2+\bar{n}}{2}\right) \exp\left(-\frac{\bar{n}}{2} + 2\sqrt{\frac{\bar{n}}{2}}x\right) \\
&= \frac{1}{\sqrt[4]{\pi}} \exp(-\bar{n}) \exp\left(-\frac{x^2}{2} + \sqrt{2\bar{n}}x\right) = \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{(x-\sqrt{2\bar{n}}\,)^2}{2}\right)
\end{aligned} \tag{10}$$

for $\bar{n} \geqslant 0$. The ambiguity function corresponding to the state $\psi(x)$, equation (10), is given by [9]:

$$W_S[\psi](p,x) = \frac{1}{2\pi} \exp(-\mathrm{i}\sqrt{2\bar{n}}p) \exp\left(-\frac{x^2+p^2}{4}\right). \tag{11}$$

Examples 1–3 presented below are based on formula (11) and its simple modifications. They allow us to illustrate the main result of this paper avoiding technical difficulties in calculation of corresponding integrals. We shall only use the formula: $\int_{\mathbb{R}} \mathrm{d}x \exp(-a(x+\mathrm{i}b)^2) = \sqrt{\pi/a}$, valid for $a > 0$ and $b \in \mathbb{R}$.

Example 1.

Let us assume that for $\bar{n} \geqslant 0$ we have a function A defined on $\mathbb{R}^2$ by the formula

$$A : \mathbb{R}^2 \ni (p, x) \longrightarrow A(p, x) := \frac{1}{2\pi} \exp(-\mathrm{i}\sqrt{2\bar{n}}\,p) \exp\left(-\frac{x^2 + p^2}{4}\right) \in \mathbb{C} \tag{12}$$

and we want to check whether A is an ambiguity function corresponding to some wavefunction and if so to find this wavefunction. First we shall check whether all conditions from fact 3 hold for A defined above, as these conditions are usually much easier to verify than the fundamental condition, equation (4), itself. If any of the conditions constituting fact 3 do not hold for A, then A is not an ambiguity function of a pure state.

As $A \in \mathcal{S}(\mathbb{R}^2)$, it follows that $A \in L^2(\mathbb{R}^2)$, and conditions 7°–10° from fact 3 are fulfilled. Because $A(0, 0) > 0$, it is enough to check point 6° from 3° to 6°, which for our example is quite straightforward. Also condition 2° is fulfilled because

$$\|A\|_2^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}p \exp\left(-\frac{x^2 + p^2}{2}\right) = \frac{1}{2\pi}.$$

Condition 1° holds because for all $(p_0, x_0) \in \mathbb{R}$, we have $|A(p_0, x_0)|^2 = \frac{1}{4\pi^2} \exp\left(-\frac{x_0^2+p_0^2}{2}\right)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\mathrm{d}p'\,\mathrm{d}x'}{2\pi} \exp(\mathrm{i}p_0x' - \mathrm{i}x_0p')|A(p', x')|^2 \\
&\quad= \frac{1}{8\pi^3} \int_{\mathbb{R}^2} \mathrm{d}p'\,\mathrm{d}x' \exp(\mathrm{i}p_0x' - \mathrm{i}x_0p') \exp\left(-\frac{x'^2 + p'^2}{2}\right) \\
&\quad= \frac{1}{4\pi^2} \exp\left(-\frac{x_0^2 + p_0^2}{2}\right).
\end{aligned}$$

Function A fulfils all conditions from fact 3, so we have to check the fundamental condition from definition 1. The left-hand side of equation (4) for any $p, p_0, x, x_0 \in \mathbb{R}$ equals

$$\begin{aligned}
&A^\star(p_0 - p, x_0 - x)A(p_0 + p, x_0 + x) \\
&\quad= \frac{1}{4\pi^2} \exp(-\mathrm{i}2\sqrt{2\bar{n}}\,p) \exp\left(-\frac{p^2 + p_0^2 + x^2 + x_0^2}{2}\right),
\end{aligned}$$

and the right-hand side of equation (4) reads

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\mathrm{d}p'\,\mathrm{d}x'}{2\pi} \exp(\mathrm{i}p_0x' - \mathrm{i}x_0p')A^\star(p' - p, x' - x)A(p' + p, x' + x) \\
&\quad= \frac{\mathrm{e}^{(-\mathrm{i}2\sqrt{2\bar{n}}\,p)}}{8\pi^3} \exp\left(-\frac{p^2 + x^2}{2}\right) \int_{\mathbb{R}^2} \mathrm{d}p'\,\mathrm{d}x'\,\mathrm{e}^{(\mathrm{i}p_0x' - \mathrm{i}x_0p')} \exp\left(-\frac{p'^2 + x'^2}{2}\right) \\
&\quad= \frac{1}{4\pi^2} \exp(-\mathrm{i}2\sqrt{2\bar{n}}\,p) \exp\left(-\frac{p^2 + p_0^2 + x^2 + x_0^2}{2}\right).
\end{aligned}$$

It is seen that function A defined by (12) fulfils the fundamental condition and $A(0, 0) = \frac{1}{2\pi}$, which (see corollary 2) means that there exists a function $\psi \in L^2(\mathbb{R})$ normalized to 1, such that $A = W_S[\psi]$. Now, we can determine this function following the steps described in the previous section.

In our case, $A(p, 0) = \frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}\sqrt{2\bar{n}}\,p}\,\mathrm{e}^{-\frac{p^2}{4}}$ and it is easy to check that

$$\int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{\mathrm{i}\sqrt{2\bar{n}}\,p} A(p, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}p\,\mathrm{e}^{-(\frac{p}{2})^2} = \frac{1}{\sqrt{\pi}} \neq 0.$$

This means that $\tau := \sqrt{2\bar{n}}$ fulfils inequality (5) and according to definition 2 let us consider the function defined by equation (6) (for simplicity of notation, we shall use $\mathcal{N}$ instead of $n_\tau = n_{\sqrt{2n}}$):

$$\begin{aligned}
\psi_{\sqrt{2\bar{n}},\mathcal{N}}(x) &:= \mathcal{N}\int_{\mathbb{R}} \mathrm{d}p \exp\left(\mathrm{i}\frac{x+\sqrt{2\bar{n}}}{2}p\right) A(p, x-\sqrt{2\bar{n}}) \\
&= \frac{\mathcal{N}}{2\pi}\int_{\mathbb{R}} \mathrm{d}p \exp\left(\mathrm{i}\frac{x+\sqrt{2\bar{n}}}{2}p\right) \mathrm{e}^{-\mathrm{i}\sqrt{2\bar{n}}p}\,\mathrm{e}^{-\frac{p^2}{4}}\,\mathrm{e}^{-\frac{(x-\sqrt{2\bar{n}})^2}{4}} \\
&= \frac{\mathcal{N}}{2\pi}\exp\left(-\frac{\left(x-\sqrt{2\bar{n}}\right)^2}{2}\right)\int_{\mathbb{R}} \mathrm{d}p \exp\left(-\left[\frac{p+\mathrm{i}(x-\sqrt{2\bar{n}})}{2}\right]^2\right) \\
&= \frac{\mathcal{N}}{\sqrt{\pi}}\exp(-\bar{n})\exp\left(-\frac{x^2}{2}+\sqrt{2\bar{n}}x\right).
\end{aligned}$$

Because

$$\left[\int_{\mathbb{R}} \mathrm{d}z\,\mathrm{e}^{\mathrm{i}\sqrt{2\bar{n}}z}A(z,0)\right]^\star = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}z \exp\left(-\frac{z^2}{4}\right) = \frac{1}{\sqrt{\pi}}$$

fact 4.3° means that $|\mathcal{N}| = \pi^{\frac{1}{4}}$. Choosing a normalization constant $\mathcal{N} = \pi^{\frac{1}{4}}$, we obtain the wavefunction given by

$$\psi_{\sqrt{2\bar{n}},\sqrt[4]{\pi}}(x) = \frac{\mathrm{e}^{-\bar{n}}}{\sqrt[4]{\pi}}\exp\left(-\frac{x^2}{2}+\sqrt{2\bar{n}}x\right).$$

This is a state whose ambiguity function is A from equation (12). Comparing this result with equation (10), we conclude that we have reconstructed the wavefunction correctly.

Next, we shall consider two additional examples based on modifications of the function A. We want to check whether they lead to ambiguity functions corresponding to pure states.

Example 2.

Let us consider function B defined for $\bar{n} \geqslant 0$ as

$$B : \mathbb{R}^2 \ni (p, x) \longrightarrow B(p, x) := \frac{1}{2\pi}\exp(-\mathrm{i}\sqrt{2\bar{n}}\,px)\exp\left(-\frac{x^2+p^2}{4}\right) \in \mathbb{C}. \tag{13}$$

Function $B \in \mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ and if it were an ambiguity function of some wavefunction $\psi \in L^2(\mathbb{R})$, it would have to fulfil conditions from fact 3. We can check that $B(0,0) = 1/(2\pi) \in \mathbb{R}$; however,

$$B^\star(-p,-x) = \frac{1}{2\pi}\exp(\mathrm{i}\sqrt{2\bar{n}}px)\exp\left(-\frac{x^2+p^2}{4}\right),$$

which, if compared with equation (13), shows that condition 6° from fact 3 does not hold for $\bar{n} > 0$. For all positive $\bar{n}$ function B is not an ambiguity function corresponding to some $\psi \in L^2(\mathbb{R})$. For $\bar{n} = 0$, B is an ambiguity function corresponding to the wavefunction $\psi(x) = \pi^{-\frac{1}{4}}\exp(-x^2/2)$.

Remark 5. Because for all $(p, x) \in \mathbb{R}^2$ functions A from example 1 and B from example 2 are connected by the following equation $|A(p,x)|^2 = |B(p,x)|^2$, the function B fulfils condition fact 3.1° but—as we know from example 2—it does not fulfil the fundamental condition from definition 1. This shows that the fundamental condition is stronger than fact 3 point 1° alone.

Example 3.

Similarly as in previous examples we want to check whether for $\bar{n} \geqslant 0$,

$$C : \mathbb{R}^2 \ni (p, x) \longrightarrow C(p, x) := \frac{1}{2\pi} \exp(-\mathrm{i}\sqrt{2\bar{n}}\,(p + x)) \exp\left(-\frac{x^2 + p^2}{4}\right) \in \mathbb{C} \tag{14}$$

is an ambiguity function of some state. First, we check whether all conditions from fact 3 are fulfilled, which similarly as in example 1 is quite straightforward. Points 7°–10° do hold, because $C \in \mathcal{S}(\mathbb{R}^2)$. By direct calculation, we also check that point 6° is fulfilled; 3°, 4° and 5° are fulfilled automatically. Points 2° and 1° can be checked using simple integrals introduced already in example 1.

For function C, fact 3 holds, so we check also the fundamental condition (4), just as in example 1. Function C defined by formula (14) fulfils the fundamental condition and $C(0, 0) = 1/(2\pi)$. Thus, corollary 2 tells us that there exists a normalized to one function $\psi \in L^2(\mathbb{R})$ such that $C = W_S[\psi]$. Let us find this function. In this case $2\pi C(p, 0) = \exp(-\mathrm{i}\sqrt{2\bar{n}}\,p - p^2/4)$ and

$$\int_{\mathbb{R}} \mathrm{d}p\, \mathrm{e}^{\mathrm{i}\sqrt{2\bar{n}}\,p} C(p, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}p\, \mathrm{e}^{-(\frac{p}{2})^2} = \frac{1}{\sqrt{\pi}} > 0.$$

It means that $\tau := \sqrt{2\bar{n}}$ satisfies inequality (5) and according to definition 2 let us consider the function (6):

$$\begin{aligned}
\psi_{\sqrt{2\bar{n}},\,\mathcal{M}}(x) &:= \mathcal{M} \int_{\mathbb{R}} \mathrm{d}p \exp\left(\mathrm{i}\frac{x + \sqrt{2\bar{n}}}{2} p\right) C(p, x - \sqrt{2\bar{n}}) \\
&= \frac{\mathcal{M}}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \exp\left(\mathrm{i}\frac{x + \sqrt{2\bar{n}}}{2} p\right) \mathrm{e}^{-\mathrm{i}\sqrt{2\bar{n}}(p + x - \sqrt{2\bar{n}})}\, \mathrm{e}^{-\frac{p^2}{4}}\, \mathrm{e}^{-\frac{(x - \sqrt{2\bar{n}})^2}{4}} \\
&= \frac{\mathcal{M}}{2\pi} \mathrm{e}^{-\mathrm{i}\sqrt{2\bar{n}}(x - \sqrt{2\bar{n}})}\, \mathrm{e}^{-\frac{\left(x - \sqrt{2\bar{n}}\right)^2}{2}} \int_{\mathbb{R}} \mathrm{d}p \exp\left(-\left[\frac{p + \mathrm{i}(x - \sqrt{2\bar{n}})}{2}\right]^2\right) \\
&= \frac{\mathcal{M}}{\sqrt{\pi}} \exp(-\bar{n}) \exp(-\mathrm{i}\sqrt{2\bar{n}}(x - \sqrt{2\bar{n}})) \exp\left(-\frac{x^2}{2} + \sqrt{2\bar{n}}x\right) \\
&= \frac{\mathcal{M}}{\sqrt{\pi}} \exp(2\mathrm{i}\bar{n}) \exp(-\bar{n}) \exp\left(-\frac{x^2}{2} + \sqrt{2\bar{n}}(1 - \mathrm{i})x\right).
\end{aligned}$$

According to fact 4.3°, the normalizing constant fulfils the equality $|\mathcal{M}| = \pi^{\frac{1}{4}}$. Choosing $\mathcal{M} = \pi^{\frac{1}{4}} \exp(-2\mathrm{i}\bar{n})$, we obtain the following wavefunction:

$$\psi_{\sqrt{2\bar{n}},\,\sqrt[4]{\pi}\exp(-2\mathrm{i}\bar{n})}(x) = \frac{\mathrm{e}^{-\bar{n}}}{\sqrt[4]{\pi}} \exp\left(-\mathrm{i}\sqrt{2\bar{n}}x\right) \exp\left(-\frac{x^2}{2} + \sqrt{2\bar{n}}x\right).$$

Function C defined by formula (14) is the ambiguity corresponding to the pure state $\psi_{\sqrt{2\bar{n}},\,\sqrt[4]{\pi}\exp(-2\mathrm{i}\bar{n})}(x)$ defined above.

Example 4.

Let $0 < q < 1$ and

$$D : \mathbb{R}^2 \ni (p, x) \longrightarrow D(p, x) := \frac{1}{2\pi}\left(1 - \frac{q(p^2 + x^2)}{2}\right)\exp\left(-\frac{p^2 + x^2}{4}\right) \in \mathbb{C}. \tag{15}$$

D is an ambiguity function for the mixed state obtained from a binomial distribution of Fock states for probabilities q and $N = 2$; see [9], equation (80). We start a verification whether D can be an ambiguity function for a pure state, checking properties described by fact 3. Similarly as in the previous examples $D \in \mathcal{S}(\mathbb{R}^2)$, so properties 7°–10° are fulfilled. Because $D(\mathbb{R}^2) \subseteq \mathbb{R}$ and $D(-p, -x) = D(p, x)$, property 6° also holds and conditions 3°–5° are automatically fulfilled. However, property 2° does not hold. We have $4\pi^2|D(0, 0)|^2 = 1$ and

$$\begin{aligned}
\frac{\|D\|_2^2}{2\pi} &= \int_{\mathbb{R}^2} \frac{\mathrm{d}p\,\mathrm{d}x}{2\pi}\,|D(p, x)|^2 \\
&= \frac{1}{8\pi^3}\int_{\mathbb{R}^2} \mathrm{d}p\,\mathrm{d}x\left[1 - q(x^2 + p^2) + \frac{q^2}{4}(x^2 + p^2)^2\right]\exp\left(-\frac{x^2 + p^2}{2}\right) \\
&= \frac{1}{4\pi^2}\left[\int_0^\infty \mathrm{d}r\, r\,\mathrm{e}^{-\frac{r^2}{2}} - q\int_0^\infty \mathrm{d}r\, r^3\,\mathrm{e}^{-\frac{r^2}{2}} + \frac{q^2}{4}\int_0^\infty \mathrm{d}r\, r^5\,\mathrm{e}^{-\frac{r^2}{2}}\right] \\
&= \frac{1}{4\pi^2}[1 - 2q + 2q^2] = \frac{1 + 2q(q - 1)}{4\pi^2} < \frac{1}{4\pi^2} = |D(0, 0)|^2.
\end{aligned} \tag{16}$$

From (16) it follows that the function D does not fulfil condition 1° of fact 3. It is easy to check that

$$\begin{aligned}
|D(p_0, x_0)|^2 &- \int_{\mathbb{R}^2} \frac{\mathrm{d}p'\,\mathrm{d}x'}{2\pi}\exp(\mathrm{i}p_0x' - \mathrm{i}x_0p')|D(p', x')|^2 \\
&= \frac{q(1 - q)\left[1 - \left(p_0^2 + x_0^2\right)\right]}{2\pi^2}\exp\left(-\frac{p_0^2 + x_0^2}{2}\right).
\end{aligned} \tag{17}$$

To sum up: the function D defined by formula (15) does not fulfil conditions 1° and 2° from fact 3; thus, it is not an ambiguity function of a pure state. Let us note that from equations (16) and (17), it is seen that for $q = 0$ or $q = 1$, the function given by formula (15) would fulfil fact 3. This is not surprising as for both $q = 0$ and $q = 1$ in the binomial distribution, only one term is left, and then the state given by formula (80) from [9] corresponds to a pure state.

5. Concluding remarks

We have derived the necessary and sufficient condition that determines whether a given square integrable function is an ambiguity function corresponding to a pure quantum state. The fundamental condition is given by equation (4) (definition 1). However, when we are interested in checking whether a given function is a pure-state ambiguity function, it is reasonable to begin such a verification checking the properties listed in fact 3. If these properties do not hold the function we investigate is not a pure-state ambiguity function. Remark 4 presents the general form of a solution of equation (4) in a form from which it is clear that any of those solutions is up to a complex factor a pure-state ambiguity function. It follows from equation (1) and corollary 1 that any function A fulfilling the fundamental condition, equation (4), is a pure-state ambiguity function if and only if $2\pi A(0, 0) = 1$. It is worth noting that equation (6) provides a method of constructing a wavefunction corresponding to a given ambiguity function. Finally, in the last section, several examples that illustrate practical use of the main results of this paper are described.

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Appendix. Definitions and notation

$\mathcal{L}^k(\mathbb{R}^N)$ is the space of all functions integrable with kth power on $\mathbb{R}^N$, where k, N are natural numbers $(1, 2, 3\ldots)$. We say that two functions $f, g \in \mathcal{L}^k(\mathbb{R}^N)$ are equivalent if and only if they differ on a set of zero measure. $L^k(\mathbb{R}^N)$ is the Banach space of equivalence classes of functions from $\mathcal{L}^k(\mathbb{R}^N)$. The norm on $L^k(\mathbb{R}^N)$ is given by

$$\|f\|_k := \left[\int_{\mathbb{R}^N} \mathrm{d}x |f(x)|^k\right]^{\frac{1}{k}}.$$

If in this paper we write $f(\cdot)$ or $f(x)$ for $f \in L^k(\mathbb{R}^N)$, it means that we deal with a function from $\mathcal{L}^k(\mathbb{R}^N)$ that represents a class f. For more details on the theory of $\mathcal{L}^k(\mathbb{R}^N)$ and $L^k(\mathbb{R}^N)$, see [12] (chapter XIII.7, XIII.14).

The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is the linear space of all smooth and rapidly decreasing functions on $\mathbb{R}^N$, i.e. $f \in \mathcal{S}(\mathbb{R}^N)$ if and only if $f \in C^\infty(\mathbb{R}^N)$ and for any $a_1, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}^N} \left| x_1^{a_1} \cdot \ldots \cdot x_N^{a_N} \frac{\partial^{b_1+\cdots+b_N}}{\partial^{b_1} x_1 \cdot \ldots \cdot \partial^{b_N} x_N} f(x)\right| < \infty.$$

$\mathcal{S}(\mathbb{R}^N)$ is a Fréchet space and its dual space $\mathcal{S}(\mathbb{R}^N)'$ is the space of tempered distributions. More information about Schwartz spaces and tempered distributions can be found in [13] (chapter 7) or chapter XIX of [12].

$C_\infty(\mathbb{R}^N)$ is the Banach space of all continuous and vanishing at infinity functions on $\mathbb{R}^N$. It means that $f \in C_\infty(\mathbb{R}^N)$ if and only if $f \in C(\mathbb{R}^N)$ and

$$\forall \epsilon > 0\, \exists r > 0 \quad \forall\, x \in \{y \in \mathbb{R}^N : \|y\| > r\} : |f(x)| < \epsilon.$$

The norm on the space $C_\infty(\mathbb{R}^N)$ is defined by $\|f\|_\infty := \sup_{x\in\mathbb{R}^N} |f(x)|$. For details see chapter IV.5 of [14].

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